The order of the group of self-homotopy equivalence of wedge spaces

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Abstract In this paper $Aut(\Sigma X \vee \Sigma Y)^{\#}$ the order of the group of self-homotopy equivalence of wedge spaces is studied. Under the condition of reducibility, we decompose $Aut(\bigvee_{t=1}^{k} X_t)$ to the product of subgroups which generalizes the known results for $k = 2$. Then we also give the formula for $Aut(\bigvee_{t=1}^{k} \Sigma X_t)^{\#}$.

keywords homotopy equivalence, reducible, wedges.

1 Introduction

In this paper, all spaces are connected pointed CW-complexes, all maps and homotopies are based point preserving. $[X, Y]$ denotes the set of basepoint preserving homotopy classes of based maps from X to Y. $Aut(X)$ is a subset of $[X, X]$ formed by self-homotopy equivalences. The operation induced by the composition of homotopy classes makes $Aut(X)$ into a group, which is normally called the group of self-homotopy equivalences of X.

We are given a map $f : X \vee Y \to X \vee Y$, for $I, J \in \{X, Y\}$, denote $f \circ i_I$ and $p_J \circ f \circ i_I$ by f_I and f_{JI} respectively, where $i_I : I \to X \vee Y$ is a coordinate inclusion and $p_J : X \vee Y \to J$ is a coordinate project. Thus, there is $f = (f_X, f_Y)$ by the universal property of wedge spaces. The group $Aut(X \vee Y)$ is called reducible if for any$f \in Aut(X \vee Y)$ there are $f_{XX} \in Aut(X)$ and $f_{YY} \in Aut(Y)$.

Suppose that M and N are two subgroups of a group G. The set

$$M \centerdot N = \{mn | m \in M, n \in N\}$$

is called the product of subgroups M and N. Note that it is not assumed that any of M, N is normal subgroup of G, hence $M \centerdot N$ is not generally a subgroup of G. Now let

$$Aut_X(X \vee Y) = \{f \in Aut(X \vee Y) | f \circ i_X = i_X\},$$

$$Aut_Y(X \vee Y) = \{f \in Aut(X \vee Y) | f \circ i_Y = i_Y\}.$$

In [3], Yu, H.B. and Shen, W.H. showed that $Aut_X(X \vee Y)$ and $Aut_Y(X \vee Y)$ are subgroups of $Aut(X \vee Y)$ and they get the following Theorem

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Theorem 1.1. *If X and Y is simply connected and $Aut(X\vee Y)$ is reducible, then $Aut(X\vee Y) = Aut_X(X\vee Y)\,.\,Aut_Y(X\vee Y)$.*

Let $G^{\#}$ denote the order of a group G. In Section 2, we will compute $Aut(\Sigma X\vee\Sigma Y)^{\#}$, where $Aut(\Sigma X\vee\Sigma Y)$ can be reducible (Theorem 2.6). The key point is to use Hilton-Minor Theorem. In Section 3, we firstly generalize the Theorem 1.1 to Theorem 3.2 for $Aut(\bigvee_{t=1}^{k} X_t)$. Then this generalization enables us to compute $Aut(\bigvee_{t=1}^{k}\Sigma X_t)^{\#}$ (Theorem 3.11).

2 Calculation of $Aut(\Sigma X\vee\Sigma Y)^{\#}$

Lemma 2.1. *The map*

$$Aut_X(X\vee Y)\times Aut_Y(X\vee Y)\xrightarrow{\phi} Aut_X(X\vee Y)\,.\,Aut_Y(X\vee Y)$$

given by $\phi(f,g)=f\circ g$ is a bijection.

Proof. It is enough to show that $Aut_X(X\vee Y)\cap Aut_Y(X\vee Y)=(i_X,i_Y)$, where (i_X,i_Y) is the unit in the group $Aut(X\vee Y)$. It is clear by the definition of $Aut_X(X\vee Y)$ and $Aut_Y(X\vee Y)$. □

Let

$$[X,X\vee Y]_{\simeq X}=\{g\in[X,X\vee Y]\ \mid p_X\circ g\in Aut(X)\},$$
$$[Y,X\vee Y]_{\simeq Y}=\{f\in[Y,X\vee Y]\ \mid p_Y\circ f\in Aut(Y)\}.$$

By [3], we know that if $Aut(X\vee Y)$ is reducible, then

$$Aut_X(X\vee Y)=\{(g,i_Y)\mid p_X\circ g\in Aut(X)\}, \tag{1}$$
$$Aut_Y(X\vee Y)=\{(i_X,f)\mid p_Y\circ f\in Aut(Y)\}. \tag{2}$$

Thus it is easy to get the following Lemma

Lemma 2.2. *If $Aut(X\vee Y)$ is reducible and X, Y are simply connected, then the following maps*

$$[X,X\vee Y]_{\simeq X}\xrightarrow{\phi_Y} Aut_Y(X\vee Y),\quad \phi_Y(g)=(g,i_Y)$$
$$[Y,X\vee Y]_{\simeq Y}\xrightarrow{\phi_X} Aut_X(X\vee Y),\quad \phi_X(f)=(i_X,f)$$

are isomorphisms of sets.

Remark 2.3. From Proposition 2.1 of [3], for $(i_X,f)\in Aut(X\vee Y)$, there is a map $\overline{f}: Y\to X\vee Y$ such that $(i_X,\overline{f})$ is the homotopy inverse of (i_X,f). Thus, $(i_X,f)\circ\overline{f}=(i_X,f)\circ(i_X,\overline{f})\circ i_Y\simeq i_Y$. Now define $f\cdot g=(i_X,f)\circ g$, $f^{-1}=\overline{f}$, then $([Y,X\vee Y]_{\simeq Y},\cdot)$ becomes a group with unit i_Y and ϕ_X is an isomorphism of groups.

Let $B=B_1\vee B_2\vee\cdots\vee B_k$. Take abstract symbols $z_1,z_2,\cdots,z_k$. Let $(F,[,])$ be the free non-associative algebraic object (over $\mathbb{Z}$) generated by $z_1,z_2,\cdots,z_k$ with one binary operation $[,]$. M is the set of monomials in F. The weight $wt(a)$ is the number of factors in $a\in M$. We define and order a "set of basic commutators" (page 438 of [1]) $Q\subset M$ inductively as follows: For weight 1, $z_1<z_2<\cdots<z_k$; Now suppose that all

elements of weight $< w$ are defined and ordered, then an element of weight $w > 1$ is a bracket $[a, b]$ where $wt(a) + wt(b) = w, a < b$ and if $b = [c, d]$ then $c \leq a$. For $k = 2$, $Q = \{z_1 < z_2 < [z_2, [z_1, z_2]] < [z_1, [z_1, z_2]] < \cdots\}$.

Let $A \cong_S B$ denote the isomorphic of sets A and B. By the Hilton-Milnor Theorem, we get that

Theorem 2.4. *Let $B = X \vee Y$, then there is an isomorphism of sets*

$$\prod_{c \in Q} [\Sigma Y, \Sigma \wedge^c B] \cong_S [\Sigma Y, \Sigma X \vee \Sigma Y]$$

which is given by $(f_c)_{c \in Q} \mapsto \sum_{c \in Q} i_c \circ f_c$, where $\wedge^c B$ and the iterated Whitehead product $i_c \in [\Sigma \wedge^c B, \Sigma B]$ are defined on page 438 of [1], and the sum of $\sum_{c \in Q} i_c \circ f_c$ is in the order indicated by the order of Q.

Corollary 2.5. *If $Aut(\Sigma X \vee \Sigma Y)$ is reducible, then*

$$Aut_X(\Sigma X \vee \Sigma Y) \cong_S [\Sigma Y, \Sigma X \vee \Sigma Y]_{\simeq Y} \cong_S [\Sigma Y, \Sigma X] \times Aut(\Sigma Y) \times \prod_{c \in Q, wt(c) > 1} [\Sigma Y, \Sigma \wedge^c B]$$

Proof. Note that $i_{z_1} = i_{\Sigma X}, i_{z_2} = i_{\Sigma Y}$. For any $i_c = [a, b]$, $p_{\Sigma Y} \circ i_c = p_{\Sigma Y} \circ [a, b] = [p_{\Sigma Y} \circ a, p_{\Sigma Y} \circ b]$. By the definition of $i_c, c \in Q$, if $wt(c) > 1$, then there is a factor $i_{\Sigma X}$ in i_c. Hence $p_{\Sigma Y} \circ i_c = \begin{cases} 0, & \text{if } c \neq z_2; \\ i_{\Sigma Y}, & \text{if } c = z_2. \end{cases}$ So $p_Y \circ (\sum_{c \in Q} i_c \circ f_c) \in Aut(\Sigma Y)$ if and only if $f_{z_2} \in Aut(\Sigma Y)$. Now by Lemma 2.2 and Theorem 2.4, we get the isomorphisms above. $\square$

Theorem 2.6. *If $Aut(\Sigma X \vee \Sigma Y)$ is reducible, then there is an isomorphism of sets*

$$\begin{aligned} Aut(\Sigma X \vee \Sigma Y) \cong_S & \ Aut(\Sigma X) \times Aut(\Sigma Y) \times [\Sigma X, \Sigma Y] \times [\Sigma Y, \Sigma X] \times \\ & \prod_{c \in Q, wt(c) > 1} ([\Sigma X, \Sigma \wedge^c B] \times [\Sigma Y, \Sigma \wedge^c B]) \end{aligned}$$

Hence if $Aut(\Sigma X \vee \Sigma Y)^\# < \infty$, then

$$\begin{aligned} Aut(\Sigma X \vee \Sigma Y)^\# = & \ Aut(\Sigma X)^\# \cdot Aut(\Sigma Y)^\# \cdot [\Sigma X, \Sigma Y]^\# \cdot [\Sigma Y, \Sigma X]^\# \cdot \\ & \prod_{c \in Q, wt(c) > 1} ([\Sigma X, \Sigma \wedge^c B]^\# \cdot [\Sigma Y, \Sigma \wedge^c B]^\#). \end{aligned}$$

Proof. The theorem is easily obtained form Theorem 1.1, Lemma 2.1 and Corollary 2.5. $\square$

From Theorem 2.6, the following Corollary is immediately obtained.

Corollary 2.7. *Let CW_n^k be the full subcategory of homotopy category formed by $(n-1)$-connected and at most $(n+k)$-dimensional CW-complexes. Suppose $Aut(\Sigma X \vee \Sigma Y)$ is reducible where ΣX, ΣY are two objects of CW_n^k.*

(i) If $k \leq n - 2$, then $Aut(\Sigma X \vee \Sigma Y) \cong_S Aut(\Sigma X) \times Aut(\Sigma Y) \times [\Sigma X, \Sigma Y] \times [\Sigma Y, \Sigma X]$.

(ii) *If* $k \leq 2n-3$*, then* $Aut(\Sigma X \vee \Sigma Y) \cong_S Aut(\Sigma X) \times Aut(\Sigma Y) \times [\Sigma X, \Sigma Y] \times [\Sigma Y, \Sigma X] \times [\Sigma X, \Sigma X \wedge Y] \times [\Sigma Y, \Sigma X \wedge Y]$.

Example 2.8. $Aut(S^n \vee S^m)$ *for* $n > m > 1$.

Since $Hom(H_k(S^n), H_k(S^m)) = 0$ *for any* $k > 0$*,* $Aut(S^n \vee S^m)$ *is reducible [4]. By Theorem 2.6,* $Aut(S^n \vee S^m) \cong_S Aut(S^n) \times Aut(S^m) \times [S^n, S^m]$*. Since* $Aut(S^k) = \mathbb{Z}/2$ *for any* $k > 0$*, we get*

(i) $Aut(S^n \vee S^m)^\# = \infty$ *if and only if* m *is even and* $n = 2m-1$*;*

(ii) *If* m *is odd or* $n \neq 2m-1$*, then* $Aut(S^n \vee S^m)^\# = 4\pi_n(S^m)^\#$.

Example 2.9. $Aut(S^n \vee \Sigma\mathbb{R}P^2)$ *for* $n > 1$.

Since $Hom(H_k(\Sigma\mathbb{R}P^2), H_k(S^n)) = 0$ *for* $k > 0$*,* $Aut(S^n \vee \Sigma\mathbb{R}P^2)$ *is reducible.*

(i) For $n = 2$*,* S^2 *and* $\Sigma\mathbb{R}P^2$ *are two objects of* CW_2^1*. Note that* $[S^n, S^n \wedge \Sigma\mathbb{R}P^2] = 0$*. From Corollary 2.7 we have*

$$Aut(S^2 \vee \Sigma\mathbb{R}P^2) \cong_S Aut(S^2) \times Aut(\Sigma\mathbb{R}P^2) \times \pi_2(\Sigma\mathbb{R}P^2) \times [\Sigma\mathbb{R}P^2, S^2] \times [\Sigma\mathbb{R}P^2, \Sigma^2\mathbb{R}P^2].$$

It is easy to know $Aut(S^2) = \mathbb{Z}/2$ *and* $\pi_2(\Sigma\mathbb{R}P^2) \cong H_2(\Sigma\mathbb{R}P^2) \cong \mathbb{Z}/2$*. Using the Barratt-Puppe sequence for cofibration sequence* $S^2 \to S^2 \to \Sigma\mathbb{R}P^2$*, we get*

$$[\Sigma\mathbb{R}P^2, S^2] = \mathbb{Z}/2, \quad [\Sigma\mathbb{R}P^2, \Sigma^2\mathbb{R}P^2] = \mathbb{Z}/2.$$

By Theorem 4 *of [2], there is a short sequence of groups:*

$$0 \to Ext(\mathbb{Z}/2, \pi_3(\Sigma\mathbb{R}P^2)) \to Aut(\Sigma\mathbb{R}P^2) \to Aut(\mathbb{Z}/2) \to 1.$$

From [1], $\pi_3(\Sigma\mathbb{R}P^2) = \mathbb{Z}/4$*, hence* $Aut(\Sigma\mathbb{R}P^2) = \mathbb{Z}/2$*. Thus*

$$Aut(S^2 \vee \Sigma\mathbb{R}P^2)^\# = 32.$$

(ii) For $n = 3$*,* S^3 *and* $\Sigma\mathbb{R}P^2$ *are also two objects of* CW_2^1 *and since* $[\Sigma\mathbb{R}P^2, \Sigma^3\mathbb{R}P^2] = 0$*,*

$$Aut(S^3 \vee \Sigma\mathbb{R}P^2) \cong_S Aut(S^3) \times Aut(\Sigma\mathbb{R}P^2) \times \pi_3(\Sigma\mathbb{R}P^2) \times [\Sigma\mathbb{R}P^2, S^3].$$

By $[\Sigma\mathbb{R}P^2, S^3] = \mathbb{Z}/2$*,*

$$Aut(S^3 \vee \Sigma\mathbb{R}P^2)^\# = 32.$$

(iii) For $n > 3$*, then by Theorem 2.6,*

$$Aut(S^n \vee \Sigma\mathbb{R}P^2) \cong_S Aut(S^n) \times Aut(\Sigma\mathbb{R}P^2) \times \pi_n(\Sigma\mathbb{R}P^2)$$

Thus,

$$Aut(S^n \vee \Sigma\mathbb{R}P^2)^\# = 4 \cdot \pi_n(\Sigma\mathbb{R}P^2)^\#.$$

3 generalization to $Aut(\bigvee_{t=1}^{k} X_t)$ and $Aut(\bigvee_{t=1}^{k} \Sigma X_t)^{\#}$

In this section, we will generalize Theorem 1.1 to $Aut(\bigvee_{t=1}^{k} X_t), k \geq 3$, where X_t is a simply connected CW-complexes for any $1 \leq t \leq k$, then we compute $Aut(\bigvee_{t=1}^{k} \Sigma X_t)^{\#}$.

We are given a map $\bigvee_{t=1}^{k} X_t \xrightarrow{f} \bigvee_{t=1}^{k} X_t$. Denote $f_i := f \circ i_{X_i}$ and $f_{ji} := p_{X_j} \circ f \circ i_{X_i}$, where $i_{X_i} : X_i \to \bigvee_{t=1}^{k} X_t$ is a coordinate inclusion and $p_{X_j} : \bigvee_{t=1}^{k} X_t \to X_j$ is a coordinate project. We have $f = (f_t)_{t=1}^{k}$ by the universal property of wedge spaces.

Definition 3.1. The group $Aut(\bigvee_{t=1}^{k} X_t)$ is called reducible if for any $f \in Aut(\bigvee_{t=1}^{k} X_t)$ and any $i \in \{1, 2, \cdots, k\}$, $f_{ii} \in Aut(X_i)$.

Let $Aut_{/X_i}(\bigvee_{t=1}^{k} X_t) = \{f \in Aut(\bigvee_{t=1}^{k} X_t) \mid f_t = i_{X_t} \text{ for any } t \neq i\}$.

Theorem 3.2. *If $Aut(\bigvee_{t=1}^{k} X_t)$ is reducible and every X_t is simply connected, then*

$$Aut(\bigvee_{t=1}^{k} X_t) = Aut_{/X_1}(\bigvee_{t=1}^{k} X_t) \centerdot Aut_{/X_2}(\bigvee_{t=1}^{k} X_t) \centerdot \cdots \centerdot Aut_{/X_k}(\bigvee_{t=1}^{k} X_t).$$

The following Proposition 3.3, Proposition 3.4, Proposition 3.5 and Proposition 3.6 can be obtained from [3] by considering

$$f = (i_{X_1}, \cdots, i_{X_{j-1}}, f_j, i_{X_{j+1}}, \cdots, i_{X_k}) \in [\bigvee_{t=1}^{k} X_t, \bigvee_{t=1}^{k} X_t], \quad 1 \leq j \leq k$$

as $(f_j, i_Y) \in [X_j \vee Y, X_j \vee Y]$, where $Y = \bigvee_{t=1, t\neq j}^{k} X_t$.

Proposition 3.3. *For $f = (i_{X_1}, \cdots, i_{X_{j-1}}, f_j, i_{X_{j+1}}, \cdots, i_{X_k}) \in Aut(\bigvee_{t=1}^{k} X_t)$ $(1 \leq j \leq k)$, then there is a map $\overline{f_j} : X_j \to \bigvee_{t=1}^{k} X_t$ such that $\overline{f} = (i_{X_1}, \cdots, i_{X_{j-1}}, \overline{f_j}, i_{X_{j+1}}, \cdots, i_{X_k})$ is the homotopy inverse of f.*

Proposition 3.4. *For any $j \in \{1, \cdots, k\}$, $Aut_{/X_j}(\bigvee_{t=1}^{k} X_t)$ is a subgroup of $Aut(\bigvee_{t=1}^{k} X_t)$.*

Proposition 3.5. *X_t is simply connected for any $t \in \{1, \cdots, k\}$, then $f = (i_{X_1}, \cdots, i_{X_{j-1}}, f_j, i_{X_{j+1}}, \cdots, i_{X_k})$ is in $Aut_{/X_j}(\bigvee_{t=1}^{k} X_t)$ if and only if $p_{X_j} \circ f_j \in Aut(X_j)$.*

Proposition 3.6. *If $Aut(\bigvee_{t=1}^{k} X_t)$ is reducible, every X_t is simply connected and $f = (f_t)_{t=1}^{k} \in Aut(\bigvee_{t=1}^{k} X_t)$, then for any $j \in \{1, \cdots, k\}$,*

$$(i_{X_1}, \cdots, i_{X_{j-1}}, f_j, i_{X_{j+1}}, \cdots, i_{X_k}) \in Aut_{/X_j}(\bigvee_{t=1}^{k} X_t).$$

Proof of Theorem 3.2. Suppose $f = (f_t)_{t=1}^{k} \in Aut(\bigvee_{t=1}^{k} X_t)$. Since $Aut(\bigvee_{t=1}^{k} X_t)$ can be reducible, by Proposition 3.6, for any $j \in \{1, \cdots, k\}$,

$${}^{j}f := (i_{X_1}, \cdots, i_{X_{j-1}}, f_j, i_{X_{j+1}}, \cdots, i_{X_k}) \in Aut_{/X_j}(\bigvee_{t=1}^{k} X_t).$$

By Proposition 3.3, there is a homotopy inverse of ${}^{1}f$ with the form

$$\overline{{}^{1}f} := (\overline{f_1}, i_{X_2}, \cdots, i_{X_k}) \in Aut_{/X_1}(\bigvee_{t=1}^{k} X_t)$$

where $\overline{f_1} : X_1 \to \bigvee_{t=1}^{k} X_t$. Hence $p_{X_1} \circ \overline{f_1} \in Aut(X_1)$. For any l with $k \geq l \geq 2$,

$$\overline{{}^{1}f} \circ {}^{l}f = (\overline{f_j}, i_{X_2}, \cdots, i_{X_{l-1}}, \overline{{}^{1}f} \circ f_l, i_{X_{l+1}}, \cdots, i_{X_k}) \in Aut(\bigvee_{t=1}^{k} X_t).$$

By Proposition 3.6, ${}^{l}h := (i_{X_1}, i_{X_2}, \cdots, i_{X_{l-1}}, \overline{{}^{1}f} \circ f_l, i_{X_{l+1}}, \cdots, i_{X_k}) \in Aut_{/X_l}(\bigvee_{t=1}^{k} X_t)$. Now we can easily see that $f = {}^{1}f \circ {}^{2}h \circ {}^{3}h \circ \cdots \circ {}^{k}h$. The proof is finished. $\square$

Note that if $\{i_1, i_2, \cdots, i_r\} \bigcap \{j_1, j_2, \cdots, j_s\} = \emptyset$, then

$$Aut_{/X_{i_1}}(\bigvee_{t=1}^{k} X_t) \centerdot \cdots \centerdot Aut_{/X_{i_r}}(\bigvee_{t=1}^{k} X_t) \bigcap Aut_{/X_{j_1}}(\bigvee_{t=1}^{k} X_t) \centerdot \cdots \centerdot Aut_{/X_{j_s}}(\bigvee_{t=1}^{k} X_t) = \{id\}.$$

Hence the following Lemma 3.7 which is similar to Lemma 2.1 is obtained.

Lemma 3.7. *The map*

$$\prod_{r=1}^{k} Aut_{/X_r}(\bigvee_{t=1}^{k} X_t) \xrightarrow{\phi} Aut_{/X_1}(\bigvee_{t=1}^{k} X_t) \centerdot Aut_{/X_2}(\bigvee_{t=1}^{k} X_t) \centerdot \cdots \centerdot Aut_{/X_k}(\bigvee_{t=1}^{k} X_t)$$

given by $\phi((f_r)_{r=1}^{k}) = f_1 \circ f_2 \circ \cdots \circ f_k$ is a bijection.

Now, the following Lemmas, Propositions, and Theorems in the rest of the paper are easily generalized from the case $k = 2$ in Section 2. We will omit the proof of them.

Lemma 3.8. *If $Aut(\bigvee_{t=1}^{k} X_t)$ is reducible and every X_t is simply connected, then the following map*

$$[X_j, \bigvee_{t=1}^{k} X_t]_{\simeq X_j} \xrightarrow{\phi_j} Aut_{/X_j}(\bigvee_{t=1}^{k} X_t)$$

$$\phi_j(f) = (i_{X_1}, \cdots, i_{X_{j-1}}, f, i_{X_{j+1}}, \cdots, i_{X_k})$$

is an isomorphism of sets for any $j \in \{1, 2, \cdots, k\}$, where

$$[X_j, \bigvee_{t=1}^{k} X_t]_{\simeq X_j} = \{f \in [X_j, \bigvee_{t=1}^{k} X_t] \ \mid p_{X_j} \circ f \in Aut(X_j)\}.$$

Theorem 3.9. *Let $B = X_1 \vee X_2 \vee \cdots \vee X_k$, for any $j \in \{1, \cdots, k\}$, there is an isomorphism of sets*

$$\prod_{c \in Q} [\Sigma X_j, \Sigma \wedge^c B] \cong_S [\Sigma X_j, \Sigma B]$$

which is given by $(f_c)_{c\in Q} \mapsto \sum_{c \in Q} i_c \circ f_c$, where the "set of basic commutators" Q, $\wedge^c B$ and the iterated Whitehead product $i_c \in [\Sigma \wedge^c B, \Sigma B]$ are defined on page 438 of [1], and the sum of $\sum_{c \in Q} i_c \circ f_c$ is in the order indicated by the order of Q.

Corollary 3.10. *If $Aut(\bigvee_{t=1}^{k} \Sigma X_t)$ is reducible, then for any $j \in \{1, \cdots, k\}$,*

$$Aut_{/\Sigma X_j}(\bigvee_{t=1}^{k} \Sigma X_t) \cong_S [\Sigma X_j, \bigvee_{t=1}^{k} \Sigma X_t]_{\simeq \Sigma X_j} \cong_S Aut(\Sigma X_j) \times \prod_{c \in Q, c \neq z_j} [\Sigma X_j, \Sigma \wedge^c B]$$

Theorem 3.11. *If $Aut(\bigvee_{t=1}^{k} \Sigma X_t)$ is reducible, then there is an isomorphism of sets*

$$Aut(\bigvee_{t=1}^{k} \Sigma X_t) \cong_S \prod_{t=1}^{k} Aut(\Sigma X_t) \times \prod_{1 \leq r < s \leq k} [\Sigma X_r, \Sigma X_s] \times \prod_{c \in Q, wt(c) \geq 2, j \in \{1 \cdots k\}} [\Sigma X_j, \Sigma \wedge^c B].$$

Hence if $Aut(\bigvee_{t=1}^{k} \Sigma X_t)^{\#} < \infty$, then

$$Aut(\bigvee_{t=1}^{k} \Sigma X_t)^{\#} = \prod_{t=1}^{k} Aut(\Sigma X_t)^{\#} \cdot \prod_{1 \leq r < s \leq k} [\Sigma X_r, \Sigma X_s]^{\#} \cdot \prod_{c \in Q, wt(c) \geq 2, j \in \{1 \cdots k\}} [\Sigma X_j, \Sigma \wedge^c B]^{\#}.$$

Example 3.12. $Aut(S^n \vee S^m \vee S^l), n > m > l > 1$

By considering the homology groups, it is easy to get the reducibility of $Aut(S^n \vee S^m \vee S^l)$ for $n > m > l > 1$.

(i) If $l+m > n+1$, then $[S^n, S^m \wedge S^{l-1}] = 0$. By Theorem 3.11,

$$Aut(S^n \vee S^m \vee S^l) \cong_S Aut(S^n) \times Aut(S^m) \times Aut(S^l) \times \pi_n(S^m) \times \pi_n(S^l).$$

$$Aut(S^n \vee S^m \vee S^l)^{\#} = 8 \cdot \pi_n(S^m)^{\#} \cdot \pi_n(S^l)^{\#}$$

(ii) If $min\{2m+l-1, 2l+m-1\} > n+1$, then by Theorem 3.11, we have

$$Aut(S^n \vee S^m \vee S^l) \cong_S Aut(S^n) \times Aut(S^m) \times Aut(S^l) \times \pi_n(S^m) \times \pi_n(S^l) \times \pi_n(S^{l+m-1}).$$

$$Aut(S^n \vee S^m \vee S^l)^{\#} = 8 \cdot \pi_n(S^m)^{\#} \cdot \pi_n(S^l)^{\#} \cdot \pi_n(S^{l+m-1})^{\#}.$$

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